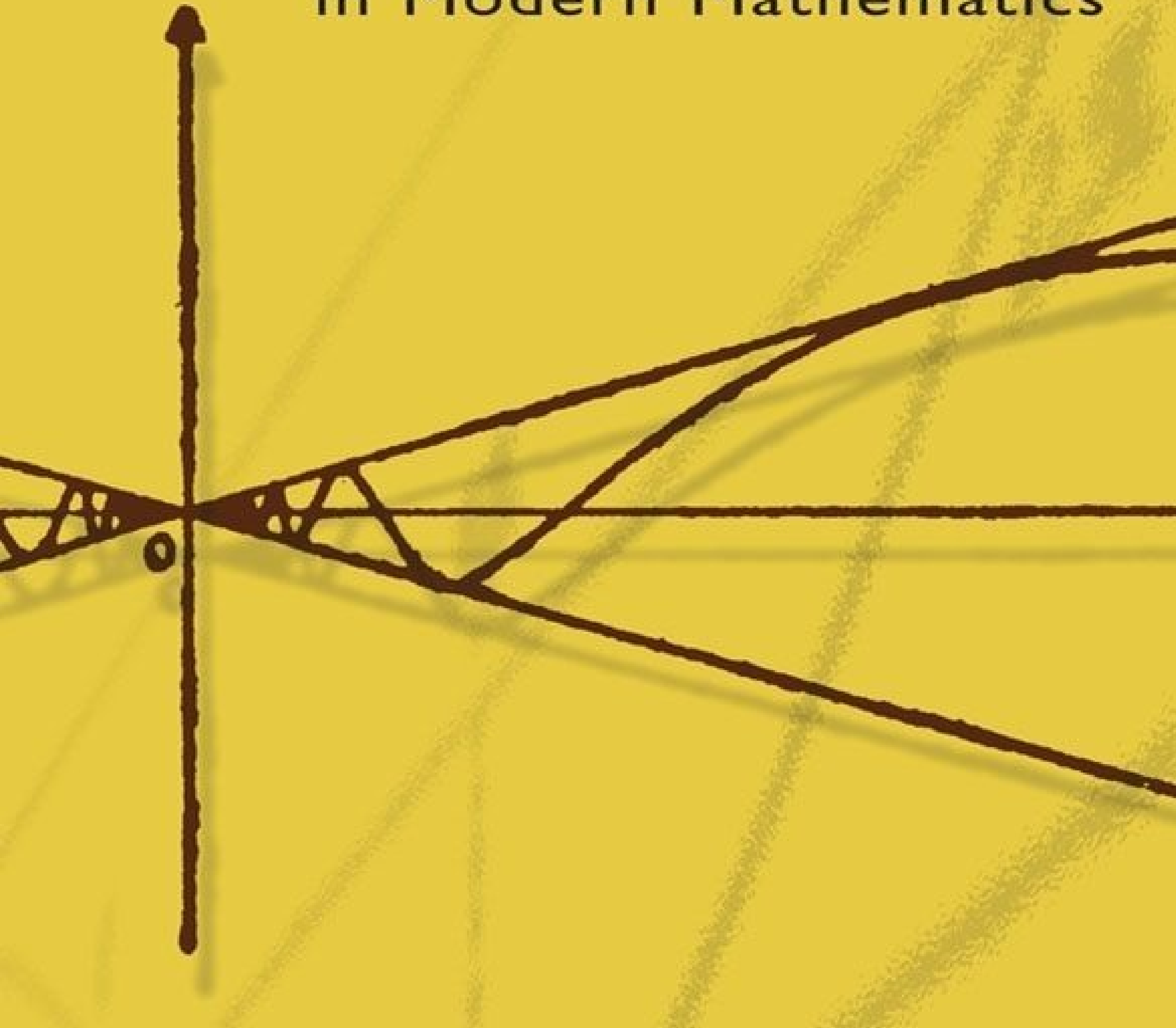
FRIEDRICH WAISMANN

Introduction to Mathematical Thinking

The Formation of Concepts in Modern Mathematics



Introduction to
MATHEMATICAL THINKING
The Formation of Concepts in Modern Mathematics

Friedrich Waismann

With a Foreword by
Karl Menger

Translated from the German
by Theodore J. Benac

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Foreword

Two methods are used to spread scientific knowledge among a larger circle of readers. One kind of popularizing deals with descriptions of the problems, and the external facts that have led to their statement and solution. With the real difficulties of the subject matter, however, it deals as briefly as possible, and at best leaves the reader to surmise its essence through ingenious comparisons. The other method refuses to bypass essential difficulties and even strives to bring those very difficulties closer to the reader. Only one course achieves this goal, and that is absolute clarity—which is frequently not found even in some original articles written for a restricted circle of readers.

To follow this second course to the end makes great demands on the author. Even the most ingenious comparisons and the most brilliant remarks in popular presentations of the first kind fail only too often to remove the doubts of the expert into whose hands they fall as to whether the author himself completely grasps the subject. Rather, absolute clarity is possible only to the scientific writer who has really penetrated the subject under consideration.

To be sure, popular books of the second group also place somewhat higher demands on the reader. Under the guidance of the author, he must reason through many sequences of unusual conceptions which he may at first find a little troublesome. But then he can be sure he will not be led to superficial semi-understanding, but to real insight.

I find it most welcome that in the field of the most modern mathematics the author of this volume has used his pedagogic skill to write a popular presentation introducing the reader to that method of thinking which actually guides the creative scholar in important parts of this science. The reader of this book will find no anecdotes connected with the fringes of scholarly mathematical work; nor will he get a cursory survey of a thousand more or less important problems of interest to the mathematician. He will, however, gain a fundamental insight into the methods of dealing with some very basic questions, above all such that are of interest to the philosopher. This insight will perhaps at first be acquired only with some difficulty. At any rate, because of the clarity of the presentation, this difficulty will be kept to a minimum.

Over and above this, questions of mathematical philosophy are also considered in the text. The author touches upon fields concerning which the most prominent scholars have held sharply divergent opinions even up to the present.

Do mathematical propositions have an empirical origin, as Mill and Mach held? Shall we believe Kant, who declared the arithmetical and geometrical propositions to be synthetic a priori judgments? Was Poincaré correct when he said that the basic rules of arithmetic were certainly synthetic a priori judgments, but that geometrical propositions were analytic—or Frege, who held that the basic rules of arithmetic were analytic and geometrical truths synthetic? Or can we finally follow those who, as in the case of Russell, characterize all mathematical propositions as analytic? Are mathematical propositions vouched for by experience? In the last analysis, do they rest on intuition and experiences of evidence? Are they founded on the fact that mathematics is a part of logic and that the latter, as is frequently said today, is a system of tautologies? Or does the foundation of mathematics rest on the proof of its consistency?

For my part, I believe that none of these questions are to be answered affirmatively. What the mathematician does is nothing but deduce statements with the help of certain methods to be enumerated, and selectable in various ways, from certain statements to be enumerated, and selectable in various ways—and all that mathematics and logic can state about the mathematician's activities which can neither supply nor requires a "foundation," is contained in this simple statement of fact. The basic approach of this book is along similar lines, which also presents new thoughts about some questions of mathematical philosophy. It is quite clear that it may hardly be possible to find undivided assent in this domain, in view of the differences of opinion described above. The reader may find much that is stimulating even on points on which, perhaps, he does not entirely agree with the text.[1]

Mathematics is used in theoretical physics and in many branches of technical science, recently also in branches of biology and economics. Statements will, therefore, frequently be brought forward in a form so general and concise that it is absolutely necessary thoroughly to understand the practice of the mathematical deduction of statements from statements if one wishes to follow through the formulation of individual statements and the combination of various statements, and especially if one wishes to proceed from starting propositions to conclusions.

But the knowledge of mathematical methods would also, I believe, be of great value even in sciences where the situation is different, as, for example, jurisprudence, sociology, and those branches of economics in which practice in the various special methods of mathematical deduction is not necessary. Indeed, in practically every discussion on any subject at all there are occasions for making use of the aforementioned insights. This does not mean that by the wider dissemination of insight into the methods of mathematics more intelligent things would necessarily be said than are said today, but surely fewer unintelligent things would be said.

It is rather unimportant which branch of mathematics is studied for the purpose of this theoretical propaedeutic, whether it be arithmetic or algebra, analytical geometry or axiomatics of elementary geometry, set theory or modern logic. What matters is that the book or lecture does not entirely neglect general methodology. Textbooks of logic, totally untouched by the modern development of this science, as they are used today for philosophic propaedeutics, are, to be sure, unsuited for such an introduction to mathematical and logical methods. Indeed, mathematical instruction is so specialized that even in many advanced mathematical textbooks and lectures it neglects the basic viewpoint, from which even the nonmathematician could gain so much profit.

Yet this methodical point of view is precisely what is brought to the forefront in this volume. And therefore wide distribution of this book will surely be of great value in many respects.

Karl Menger

Author’s Preface

It is the aim of these reflections to give an insight into the nature of mathematical concept formation, that is, to point out in the activities of the mathematician what might be of interest to a philosophically minded observer. This immediately differentiates the volume from textbooks of mathematics. The reader will not find here a system of theorems with completely developed proofs. He will find no calculations of examples nor applications of mathematics. All that is pushed into the background in favor of a presentation of mathematical ideas.

In the first place, we shall be concerned here in greater detail with the structure of the realm of numbers. The choice of this topic needs a brief justification. Proceeding from intuitive points of view, Leibniz and Newton created differential and integral calculus. In the eighteenth century these investigations soared extraordinarily, one brilliant discovery following another in the sphere of pure analysis as well as in the domain of their applications. This period of mathematics has been compared, not unjustly, with the period of the great discoverers and the heroes of the sea. The mathematicians of that age had the feeling of stepping into a new intellectual world, eager to explore the contours of the continent that sprang up before them out of the mist. In odd contrast to this series of wonderful discoveries was the obscurity that spread over the foundations of the entire concept creation. It cannot be maintained that Leibniz and Newton were very clear about the meaning of a differential quotient. Their expositions vary, but on the whole they had a dim notion of calculating with infinitely small quantities. What this means is difficult to say; and so a certain obscurity has been connected with “infinitesimal calculus” since its birth. Clear-thinking minds, such as the philosopher Berkeley, have not been sparing with their criticism; in the treatise “The Analyst” (1737) the reader finds a very detailed discussion of the new science, a discussion that turned out to be rather crushing. A remark by Lagrange, who lived at the end of the eighteenth century, attests to the fact that it was not only philosophers who had such thoughts but that even the mathematicians did not feel very comfortable in their activities. He found that the condition of mathematics was really deplorable; that it swarmed with contradictions; and that if, in spite of all this, it had achieved such great success, it was only because God in his infinite goodness had ordained that the errors cancel one another out.

No wonder this calculus appeared as something mysterious, almost as something mystical, an art more than a science, prompted by inspiration but not accessible to logical thinking. This view has even infiltrated into textbooks. For example, in Lübsen, an author well known in the last century, we read that differential calculus is a mystical method operating with infinitely small quantities; the differential is a breath, a nothing. Then follows an English quotation: an infinitesimal is the spirit of a departed quantity.

This view has lived on in the general public to the present day, and it has given rise to many an odd idea. An example of this is the well-known book of Vaihinger, “Die Philosophic des Als-Ob (The Philosophy of As-If),” in which the opinion is advanced that our theoretical thinking is frequently guided by fictions, that is, by wittingly false assumptions, which, however, have held up by their results. Vaihinger regarded differential and integral calculus as a mainstay of this view, for he thought that their basic concepts had an entirely fictitious nature. It is significant that all the authors whom Vaihinger summons regarding his view are, in every case, mathematicians of the seventeenth and eighteenth centuries, that is, men who could not have any knowledge of modern ideas.

In reality, the first half of the nineteenth century had already brought some light into this darkness; we mention here only Gauss, Cauchy, and Bolzano. These investigators prepared the way for the new critical period of mathematics, which, much more than previous periods, insists on clear definitions of concepts and logical rigor of proofs. Their work was continued and, in a certain sense, completed by Weierstrass, Cantor, and Dedekind. In the investigations of these scholars it now turned out that the real root of the difficulties lies in a clear comprehension of the concept of continuum. This concept is very closely connected with the concept of irrational number; and so we understand why these investigators were finally led to the examination of the number concept. Since the lectures of Weierstrass, it has been customary to begin a rigorous presentation of differential and integral calculus with a discussion of the number concept.

We, too, will take this course in order to become acquainted with the most important concepts of present-day mathematics.

F. W.

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1. The Various Types of Numbers

The numbers presented to us at the first stage of development are the natural or cardinal numbers 1, 2, 3, 4 . . .; they are used for counting purposes. Numbers are usually represented geometrically as points on a straight line. We will frequently employ this technique to make the following investigations clearer. For this purpose we choose an arbitrary point on a line as the starting point, also an arbitrary interval as the unit of length, and then successively mark off this interval in one direction. The numbers 0, 1, 2, 3 . . . are assigned to the points thereby generated.

Figure 1

These points are now the “images” of the numbers, and it is advantageous for many purposes to tie our concepts to this scale of points. Henceforth we will speak of the number series also as a point series.

What properties belong to the system of natural numbers?

1. It is an *ordered system.* This means that if two distinct natural numbers are given, one must precede the other; in other words: the relations a > b, a = b, a < b (a greater than b, a equal to b, a smaller than b) form a complete disjunction.

2. Consequently the concept of “betweenness” can be applied to natural numbers; that is, to say that the number c lies between a and b, implies a > c > b or a < c < b. On examining the natural numbers with respect to this concept, we encounter a rather characteristic property: every number lies between two others, its immediate predecessor and its immediate consequent. *Between two numbers immediately following one another no further number can be inserted.*

3. There is only one exception. The number 0 *does not have a predecessor.* On the other hand, there is no number which does not have a consequent. We will express these facts as follows: the number series has a first but not a last element; or: it is *infinite. on one side.*

The possibility of mapping numbers on the points of a line rests on the fact that the above properties can be ascribed to the point series. Thus it is ordered as soon as we run through the points, let us say, from left to right and think of those points which lie further to the left as antecedent. The other quoted properties also apply. Hence the structure of the number system can be carried over to that of the point system.

We obtain two further properties of the system of natural numbers as soon as we take the arithmetical operations into consideration. Which of the four basic rules of arithmetic (addition, subtraction, multiplication, division) can be performed unlimitedly in this domain, so that the result is always a natural number? Obviously only two—addition and multiplication. In contrast, the subtraction a—b can be carried out only if the minuend a is greater than the subtrahend b; and the division without a remainder only if the dividend is a multiple of the divisor.

If we combine the numbers of the domain arbitrarily by addition and multiplication, we never leave the domain. The natural numbers appear in this respect as a totality, a closed system. We will state these facts as follows: the domain of the natural numbers is closed under addition and multiplication, not closed under the two other arithmetical operations. This latter fact was actually the reason for extending the system of natural numbers in two directions. First we have to introduce the negative and secondly the fractional numbers in order to settle the closure of the number domain. Let us now take a closer look at these number creations.

The negative numbers can be thought of as generated by reversing the formation rule of the number series. Thus, instead of successively adding the number 1, we descend from the number 3 to the number 2, from 2 to 1, from 1 to 0, and then to numbers which we designate sequentially by—1,—2,—3, etc. These are represented by points as indicated below:

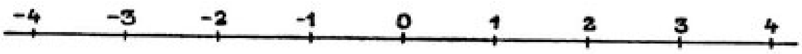

Figure 2

The system of positive and negative numbers is called the system of *integers.*

Let us now compare the integers with the natural numbers. What properties remain intact under this extension? The integers are also an ordered system; therefore the concept “betweenness” has a meaning; however, there is now no longer a number which precedes all others; the system has neither a first nor a last element; it is *infinite on both sides.* Moreover, three operations can be performed unlimitedly—not only addition and multiplication but also subtraction. However, in general, the quotient of two integers is not an integer.

The *fractional numbers* must be introduced if division is to be performed unlimitedly. The system of integers and fractional numbers is called the system of *rational numbers.* This system is closed under all four arithmetical operations. Hence we always stay within the system whenever we combine the individual elements by the four operations. A domain with this property is also called a “field” (the word is used here in a technical sense, say as in “battlefield” or in “field of force”). The system of rational numbers is ordered; for if two unequal fractions are given, one must be greater or smaller than the other. Moreover, we can think of the integers as written formally as fractions. If we try to represent the rational numbers according to the earlier model, as points on a line, we are confronted with a characteristic difficulty. Obviously the fractions lie between the integers, and consequently we must insert further points in the space between the equidistant points of Fig. 2. But how are these points spread out? If we start out, say, from the number ½, is there still an immediate predecessor or an immediate consequent? By no means! For if we choose a fraction which lies as close to ½ as we wish, it is a simple matter to obtain another fraction which lies still closer to ½. (The reader may prove this by showing that the number $\frac{a+c}{b+d}$ always lies between the two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$.)

The totality of rational numbers therefore possesses a structure completely different from that of the natural numbers and integers. Between two rational numbers there always exists another rational number. In order to characterize this specific structure the concept "dense" has been coined. We define an ordered system of elements as dense if between any two elements of the system there always lies another element of the system. The system of rational numbers is our first example of a dense system. The natural numbers and integers do not have this property.

The property of denseness makes it especially difficult to obtain an intuitive picture of the distribution of the rational numbers. We can certainly insert further points between those corresponding to the integers; however this process of insertion must be thought of as continued without end, so that in every interval of the number axis, no matter how small, there will lie an infinite number of rational points. In trying to visualize this system as a completed totality we are inevitably confronted with certain oddities. To illustrate, let us consider the totality of proper fractions with the exception of 0 and 1. The class of these points can obviously be placed in the interval of the number axis between 0 and 1; therefore we will think of it as a point set which covers this interval as an infinitely fine dust. This view leads to the following thought. When I run over the line, say, from the left to the right and beginning from a point to the left of 0, I must at some time or other meet a first element of the point set and when I have run through the whole interval, also a last element; the set must possess a point which is furthest to the left and another which is furthest to the right. A moment's reflection shows that this is absolutely impossible. It follows from the structure of the rational numbers that there is no smallest proper fraction (and also no greatest one). Conceptually this situation presents no difficulty whatsoever; the class of proper fractions is clearly and sharply defined; however the attempt to realize this concept as an intuitively clear picture leads to paradoxes. This illustrates the fact that, though we can learn some things about such relations by graphic methods, we will be misled if we entrust ourselves to them alone.

The stepwise extension of the number domain, which we have sketched above, is brought to some kind of a close with the rational numbers. And many will be tempted to assume that the extension can be carried no further, since the rational points fill up the number axis completely and without holes. That this is a mistake, that the rational numbers, even though sown infinitely dense, still do not cover the entire number axis, is the great discovery of Pythagoras. He first recognized that there are numbers which, though completely different from the rational numbers, are still related to them. We will illustrate this extraordinary discovery by constructing a square on an interval of length 1 and drawing a diagonal in this square. Our intuition tells us that this diagonal must have a very definite length. Let us try to compute it. According to the theorem of Pythagoras the square of the diagonal is equal to the sum of the squares of the two legs of the right triangle, therefore 2. Consequently the diagonal has a length $\sqrt{2}$; $\sqrt{2}$ is that number whose square is 2. Can it be represented as a fraction? First it is clear that it lies between 1 and 2; therefore we experiment with $1\frac{1}{2}$;the square of this number is $\frac{9}{4}$, therefore it is too large. The number under investigation must accordingly be greater than 1 but less than $1\frac{1}{2}$; $\frac{4}{3}$ is such a number; the test shows however that it is too small. Let us continue this process of inserting a fraction between those hitherto investigated and testing whether its square is exactly 2. In this way we will find numbers which are either too large or too small, and which we can arrange in the form of two sequences:

too small:	too large:
1	2
$^4/_3$	$^3/_2$
$^7/_5$	$^{10}/_7$
$^{24}/_{17}$	$^{17}/_{12}$

Now one could think that if we continue this search on and on and take all the time and effort that is needed, we must eventually arrive at a number whose square is exactly 2. This point will be clarified as soon as we have determined whether the trials attempted so far have failed due to chance or because there is a deeper reason underlying it. If there is a rational number which is exactly $\sqrt{2}$, then there is a fraction $\frac{p}{q}$ such that $\frac{p^2}{q^2} = 2$. Now a fraction is equal to an integer only if the denominator goes into the numerator without a remainder. Hence p^2 is divisible by q^2 . But this is only possible if p is also divisible by q. For, if p and q are two relatively prime numbers (i.e., two numbers which have no common prime factors), then p^2 and q^2 are also relatively prime; by the squaring process no prime factors can be generated which do not already exist. Hence if $\frac{p^2}{q^2} = 2$, that is, an integer, then $\frac{p}{q}$ must also be an integer. However this is impossible since $\frac{p}{q}$ lies between 1 and 2, an interval in which there are no integers.

Thus the simplest reflection on the divisibility properties of numbers readily shows that the attempt to find a rational number whose square is exactly 2 must be fruitless. On the other hand there can be no doubt that the diagonal of the unit square has a very definite length. If we think of this length as laid on the number axis with one end at 0, we obtain a point which is the geometrical representative of √2 ; the point where √2 lies can not be a rational point of the number axis. Hence we have the following result: Even though the rational points cover the number axis as an infinitely fine dust, they still do not completely fill it up. They form, as it were, a porous system, in which cracks and crevices leave room for another type of number, the *irrationals.*

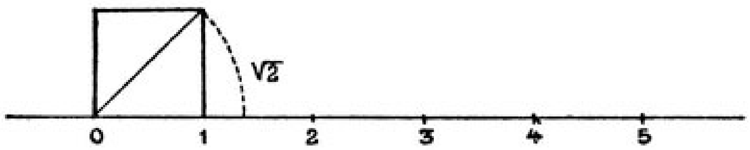

Figure 3

What can we say about the distribution of the irrational numbers? That is, are they exceptions—to be found between the rational numbers only here and there? We can answer this question by a very simple argument. Let us think of the entire number axis with the rational points on it as enlarged in the ratio 1 : √2, that is, in such a way that √2 is used as unit interval. Then every rational number will go over to a number which can be shown, as in the case of √2, to be irrational; for instance, 1 in √2, $\frac{3}{10}$ in $\frac{3}{10}\sqrt{2}$, etc. We thereby obtain a second system which is also dense, consists only of irrational numbers, and is somehow squeezed in between the rational numbers. But irrational numbers can be generated in many ways; for instance not only by square roots but also by cube roots, fourth roots, etc. In fact there are infinitely many operations which will produce irrational results when "set loose" on an individual rational number. These remarks lead us to surmise that the irrational numbers make up the principal part of the structure of the number axis instead of being its exceptional points. In the theory of sets it is actually shown that most of the points on the number axis are irrational and that the rational numbers are vanishing exceptions.

The system of rational and irrational numbers is called the system of *real numbers.*

We can also look at these relations from another point of view by proceeding from the representation of numbers as decimal fractions. A decimal fraction can terminate or continue without end. We will state, first of all, that every terminating decimal fraction can be transformed into a non-terminating one by diminishing the last numeral by unity and permitting only nines to follow afterwards. For instance, we have ½ = 0.5 = 0.4999 . . .

If we use this property, we can represent the totality of the real numbers by non-terminating decimal fractions. These fall into two categories: the periodic and non-periodic decimal fractions, the former corresponding to the rational numbers, the latter to the irrationals. (We omit the proof, which is simple.) This shows anew that between two rationals there must always lie irrational numbers; for we can always insert arbitrarily many non-periodic decimal fractions between two periodic ones.

By passing on to the imaginary numbers we take a new direction in the extension of the number concept. They were unknown in antiquity. They appeared for the first time in a work of Cardano (1545), whose name is connected with the solution of cubic equations. However, the mathematicians of that day did not have a clear understanding of the nature of these quantities. On the contrary, the imaginary numbers forced themselves into calculations against the desires and inclinations of mathematicians. This situation resulted from algorithmic requirements. Cardano's formula often represents the solution of a cubic equation, even though it may be real, in a form in which square roots of negative numbers must be extracted. Now, there is no real number whose square is negative; therefore these roots are "impossible." However, in spite of these scruples this new type of expression was manipulated like an ordinary root; and the end justified the means. We encounter here a factor which on the whole played an important role in the history of mathematics. It seems that an independent, onward-driving force is inherent in the manipulation of formulae, in the algorithms; and that in our case it induced the mathematicians to handle imaginary numbers; to the great advantage of mathematics; for the pedantic requirements of rigor would probably have paralyzed the further development. Fortunately the mathematician of that day treated subtle logical concepts with indifference; not, however, to such an extent that he would not retain, when operating with these remarkable entities, a certain uneasiness, a bad conscience, which betrayed itself in names as "impossible" or "fictitious numbers." As evidence of this, Leibniz made the following statement in the year 1702: "The imaginary numbers are a fine and wonderful refuge of the Divine Spirit, almost an amphibian between being and non-being." We note here a reflection of the strange impression which these numbers must have made on the mathematician. Thus we find that Euler was candidly astonished by the remarkable fact that a number as $\sqrt{-1}$ is neither smaller nor greater than 5, neither positive nor negative, and that it cannot be compared with ordinary numbers at all. And when a student hears about imaginary numbers for the first time, he again experiences this impression of mysteriousness, which disappears later in proportion as he learns to use these numbers. However, the nature of these numbers is not made clearer by usage. We simply have to become accustomed to them and ask nothing further. Under these circumstances it marks an epoch-making development that Gauss should give a geometrical representation of the imaginary numbers. It is found for the first time in his own abstract of a number-theoretical work in the year 1831 and has made an extraordinary deep impression. However, we know from Gauss' diary, which was left among his papers, that he was already in possession of this interpretation by 1797. Through this representation Gauss intended to clarify the "true metaphysics of imaginary numbers" and bestow on them complete franchise in mathematics. Now what is this representation? We already know that the rational and irrational numbers fill up the number axis, so that there remains not even the smallest of gaps. Hence, if we now wish to interpret the imaginary numbers geometrically, we must use a second line. In the Gaussian interpretation the real numbers are represented as points on the x-axis, the imaginaries as points on the y-axis of a rectangular Cartesian system of coordinates, whose intersection point represents the number 0. Hence a rotation through 90° takes the positive real number axis into the positive imaginary number axis. Gauss did not give a basis for this representation; however he derived from it the right to operate with imaginary numbers.

By means of this interpretation we can also obtain a geometrical picture of those numbers which are generated by the addition of an imaginary and a real number, as 2 + 3i, the so-called *complex numbers.* Such a number can be represented as the point (x = 2, y = 3) of the coordinate plane. We thereby see that the images of the complex numbers are distributed over the plane. For the representation of complex numbers a line no longer suffices. We must have recourse to a plane; the number world has been broadened to a two-dimensional manifold. This clearly shows the reader that a far-reaching step has been taken. Up to now if one has understood by a number something which could be arranged serially by "greater" and "smaller," then this is no longer valid in the domain of complex numbers. For example, which of the two numbers 2 + 3i and 3 + 2i is the greater? The (linear) order is not valid and therefore neither is the concept of "betweenness." This shows that in the transition to complex numbers it is no longer possible to compare numbers with respect to their magnitude, a property which hitherto was thought of as wholly essential for the concept of number.

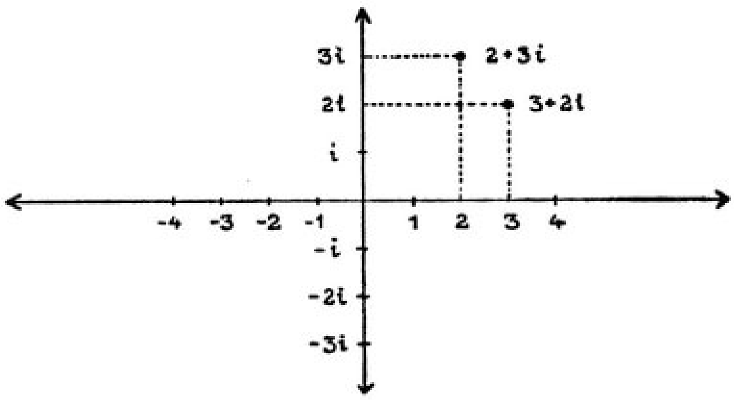

Figure 4

2. Criticism of the Extension of Numbers

In the previous chapter we presented an introductory, orienting view of the extensions of the number domain in order to acquaint ourselves as soon as possible with the subject matter under investigation. In particular we apprehended that the reason for introducing new numbers was the requirement that subtraction, division and evolution be operations which could be performed in all cases.

This is the way in which our subject matter is usually represented, as it is studied today in school and, generally speaking, as things developed historically. And yet it is easy to raise questions which perplex us from this point of view. Can we continue the extension of the number domain still further? Can we invent numbers which are no longer representable in the plane, but which require a mapping in the three-dimensional space? Or is this impossible? And on what does it essentially depend? However there is a question which is more important than these. Up to now we have said that it was the desire to make certain operations possible without any exception, which urged us to extend the number domain. Thus it was the non-applicability of subtraction which led to the introduction of negative numbers; while that of extracting roots, led to the introduction of the irrationals and later of the imaginary numbers. We could describe this as follows: the proposition to subtract 7 from 5 has no solution only as long as we restrict ourselves to the natural numbers; this does not prove that it does not have a solution in the absolute sense, but only that the domain of natural numbers is too meager to enable us to solve the proposition. Consequently we extend this number domain by annexing the negative numbers, and now the solution exists. But is this always possible? Could we solve every insolvable problem by introducing new numbers and writing the solution in terms of a new number? For instance, the operation $\frac{1}{0}$ has no solution in the usual arithmetic; for there is no number which gives 1 when multiplied by 0. Could we in this case, too, argue that only the present numbers are insufficient to solve this problem? If this point of view is valid, let us extend the domain of numbers to include a solution of this operation. Let us set $\frac{1}{0} = \omega$ and then calculate with ω as we did earlier with i. Well now, let us try to build a new arithmetic on this foundation. Is this legatimate?—The equation $1^x = 2$ has a solution neither in the domain of the real nor of the complex numbers. Could we not force the solvability by declaring that certainly there is a number which satisfies this equation? Let us consider the following two equations:

$$x + y = 10$$

$$2x + 2y = 30$$

Everyone will say that these equations do not have a solution, for the second contradicts the first. Shall we reply that this is true but only when we restrict ourselves to the numbers known up to now? What is to hinder us from inventing a new kind of number by which such a system of equations can be solved? Nothing—if the introduction of new numbers only amounts to postulating the existence of numbers which solve a stated problem. But is this actually a legitimate procedure? We certainly should not confuse wishful thinking with wish fulfillment. Consequently to wish that a number shall exist whose square is 2 or whose square is—1 is not the same as saying that such a number actually exists. "Why not also ask that a line pass through three arbitrary points? Because this condition contains a contradiction. Above all one must first prove that these other conditions do not contain contradictions. Before one has done this, all striving after rigor is nothing but mere pretense and sham." (Frege.) And Russell remarked, "the method, whereby one 'postulates' what one needs, has many advantages. They are the same as the advantages of the thief face to face with an honest task." No, in the postulational method there does not reside a secret magic power. It is an expedient which is far too evasive to be valid.

Hence we must admit that the entire present structure of the number world hangs in the air; we do not quite know whether the negative, the fractional, the irrational numbers exist;[2] we do not even know what permits us to extend the number domain. We must begin anew.

However, perhaps our criticism goes too far. An advocate of the customary interpretation could contend that entities such as negative, fractional, irrational numbers clearly exist in the various applications of arithmetic. For instance, the existence of $\sqrt{2}$, that is, the solvability of the equation $x^2—2 = 0$ follows very conclusively from the interpretation of Pythagoras, whereby the diagonal of the unit square has the length $\sqrt{2}$. Similarly, the existence of fractional numbers can also be established from a purely geometrical point of view, by dividing the unit interval into equal parts. And in the case of negative numbers we not only have their representation on the number axis but also their application to hot and cold temperatures, to assets and debits, to elevation above and depression below sea level, and so on; therefore the calculation with negative numbers has a clear meaning. Furthermore, such associations have helped the mathematician very effectively in the conception of those new ideas—therefore why should we reject this procedure?

Two points must be considered in order to estimate correctly the value of such opinions.

First, it is reasonable to require that arithmetic should be separated from its applications. There is no doubt that trains of thoughts like those mentioned above are highly suggestive, for they certainly have guided mathematicians in the conception of their ideas. Here, however, we are only concerned with the *justification* of operating with the new numbers, and from this point of view we must admit that the quoted examples are not convincing. Will anyone seriously assert that the existence of negative numbers is guaranteed by the fact that there exist in the world hot and cold, assets and debits? Shall we refer to these things in the structure of arithmetic? Who does not see that thereby an entirely foreign element enters into arithmetic, which endangers the pureness and clarity of its concepts? Even if there did not exist in the empirical world a distinction between hot and cold, assets and debits, this would not affect the right to introduce positive and negative numbers. If we were to base the existence of these numbers on such facts, then we would be tying arithmetic too closely to the accidental occurrences of the empirical world. And finally, we do not even feel satisfied by such a representation, since it is not sufficient for establishing the arithmetic of integers. This is substantiated by the fact that, in the attempt to introduce the negative numbers intuitively, the rule of signs, minus times minus = plus, forms the stumbling block; it cannot be made graphic by such a demonstration, therefore, such attempts complicate rather than clarify matters. And what shall we say about certain higher complex numbers, certain transfinite numbers, as introduced by Georg Cantor and others, which are capable of no such demonstration? Is this a reason to believe that they don't exist? Or must we first wait until someone has found an application of these numbers to things or events in reality?

Contrariwise, the requirement which states that arithmetic should find its hypotheses in itself and refer in no way whatsoever to those things that are not arithmetical—to experience, to perception, or anything else—is certainly appropriate. H. Hankel rejected every attempt of this kind with these clear words: "The condition for erecting a universal arithmetic is therefore a purely intellectual mathematics, one detached from all perceptions, a pure theory of forms, in which it is not quantity or its representatives, numbers, that are tied together, but intellectual objects, concepts, to which actual objects or relations may or may not correspond." (*Theorie der komplexen Zahlensysteme*, Theory of Complex Number Systems p. 10.)

The second remark is that even if we were to refer to geometry when introducing the irrational numbers, we could thereby only recognize the existence of those numbers which could be *constructed.* Now the concept *"constructibility"* can be classified into: constructible with ruler, with compass, with ruler and compass, or with some other kind of mechanism. This means that this concept is always to be understood relative to a set of allowable construction methods. According to the usual way of defining these methods it turns out that the points thereby constructed form only an insignificantly small minority. For instance, with ruler and compass all points on the number axis which correspond to the so-called transcendental numbers (these are numbers which do not satisfy an algebraic equation, for example, π, log 2, $2^{\sqrt{2}}$), cannot be constructed; this is also true of the majority of the irrational numbers which satisfy an algebraic equation—the so-called algebraic numbers. All these points would therefore elude us if we were to make use of such a geometric construction.

However, even for reasons of principle alone, it is not feasible to draw on space perceptions for the foundation of arithmetic. We will try to describe the reasons which forced this position on the mathematician. For this purpose we must take as general a view as possible of the relations between arithmetic and geometry. We begin with a brief description of the structure and development of geometry.

3. Arithmetic and Geometry

Geometry was first developed according to rigorous scientific principles in the στοιχεῖα of Euclid. This treatise was long recognized as a model. It starts with a few intuitively given propositions whose validity is not subject to further discussion; it then seeks to build up the structure of geometry so that all theorems follow by rigorous logical methods from these basic propositions without any further reference to intuition.[3] These basic propositions, which are assumed to be self-evident, are called *axioms.* Among the axioms of Euclid two groups can be distinguished:

1. General axioms of magnitude (Κοιναὶ Ἔννιαι), such as: “Two quantities equal to a third, are equal to one another,” “Equals added to equals give equals,” “The part is smaller than the whole.”

2. The essentially geometric axioms *(αἰτήματα).* As such Euclid laid down five propositions:

1. Every point can be joined to any other point by a straight line.
2. Every straight line can be extended beyond each of its endpoints.
3. A circle can be drawn with an arbitrary radius about any point.
4. All right angles are equal to one another. And now comes a very remarkable proposition:
5. If two lines are cut by a third so that the angles inside the two lines and on the same side of the third have a sum less than two right angles, then the two lines intersect on the specified side when sufficiently extended.

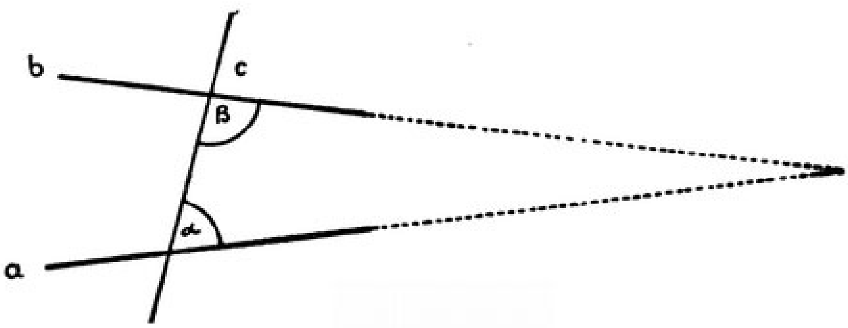

Figure 5

This last axiom, the parallel axiom, was above all the starting point of the historical development. In the presence of such a complicated proposition, the question was naturally raised as to the source of its certainty, for it is too involved to be spoken of as